

Some apriori estimates of G-BSDEs and the G-martingale representation for a special case

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Abstract This paper presents the integral(or differential) form of G-BSDEs, gives some kind of apriori estimates of their solutions, and under a very strong condition, proves the G-martingale representation theorem, and the existence and uniqueness theorem of G-BSDEs.

Keywords G-expectation; G-Brownian motion; G-BSDE; G-martingale representation theorem; existence and uniqueness

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1 Introduction

Backward stochastic differential equations(BSDEs) was first introduced by Pardoux and Peng [8] in the classical linear expectation case. Since then on, a lot of works have been devoted to study BSDE theory and its applications.

Based on BSDE theory, Peng [9] introduced the notion of g-expectation and conditional g-expectation which is the first dynamically consistent nonlinear expectation.

Then Peng [10] introduced the notion of G-expectation, which is a more general dynamically consistent nonlinear expectation, and the concept of G-Brownian motion, and then established the related stochastic calculus. The theory of G-expectation is intrinsic in the sense that it is not based on a given (linear) probability space, and it takes the probability uncertainty into consideration. Drift uncertainty and volatility uncertainty are two typical situations of probability uncertainty. G-Brownian motion has a very rich and interesting new structure which non-trivially generalizes the classical Brownian motion. G-expectation theory has developed rapidly since the initial paper Peng [10]. Peng [11, 13, 14] studied the central limit theorem under sublinear expectations and obtained that the limit distribution exists and is just the G-normal distribution. Peng [12] and Peng [15] systematically developed the stochastic calculus under G-expectation. Xu and Zhang [19] studied the Itô's integral with respect to G-martingales and the Lévy characterization of G-Brownian motion. Gao [3] studied the path properties of the solutions of G-SDEs. Hu and Peng [5] studied G-Lévy processes. Li and Peng [7] studied the Itô's integral without the condition of quasi-continuous and on stopping time interval, and generalized the Itô's formula to general $C^{1,2}$ -function. Denis, Hu and Peng [2] and Hu and Peng [5] studied the representation theorem of G-expectation and its application to G-Brownian motion paths. Soner, Touzi and Zhang [17], Song [18], Hu and Peng [6], and Peng, Song and Zhang [16] studied the G-martingale representation theorem.

G-Brownian motion has independent increments with identical G-normal distributions which means it can characterize the volatility uncertainty. And a very interesting new phenomenon is that its quadratic process generally is not a deterministic process but a stochastic

process which also has independent increments with identically maximal distributions. So the stochastic differential equations driven by G-Brownian motion(G-SDEs) of the following form

$$X_s = b(s, X_s)ds + h(s, X_s) : d\langle B \rangle_s + \theta^*(s, X_s)dB_s, t \in [0, T]$$

will carry the characteristic of both mean uncertainty and volatility uncertainty. Peng [10] proved the existence and uniqueness of the solutions of such G-SDEs. Gao [3] gave some moment estimates and Hölder continuity results of the solution of G-SDEs. However the corresponding problems for backward stochastic differential equations are not completely solved. Peng [15] give partial results to this direction, i.e., the following type of G-BSDE:

$$Y_t = \mathbb{E}[\xi + \int_t^T f(s, Y_s)ds + \int_t^T h(s, Y_s) : d\langle B \rangle_s \mid \Omega_t], t \in [0, T].$$

has a unique solution if the coefficients f, h satisfies Lipschitz condition. Hu, et al (2012) prove the existence and uniqueness result of the following G-BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s) : d\langle B \rangle_s - \int_t^T Z_s^* dB_s - (K_T - K_t)$$

by applying the partition of unity theorem to construct a new type of Galerkin approximation.

The aim of this paper is to give some kind of apriori estimates of G-BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \eta_s)ds + \int_t^T g(s, Y_s, Z_s, \eta_s) : d\langle B \rangle_s - \int_t^T Z_s^* dB_s + \int_t^T G(\eta_s)ds - \frac{1}{2} \int_t^T \eta_s : d\langle B \rangle_s.$$

or, equivalently, the differential form

$$-dY_t = f(t, Y_t, Z_t, \eta_t)dt + g(t, Y_t, Z_t, \eta_t) : d\langle B \rangle_t - Z_t^* dB_t + G(\eta_t)dt - \frac{1}{2} \eta_t : d\langle B \rangle_t, Y_T = \xi,$$

and under a very strong condition, we get the G-martingale representation theorem, and the existence and uniqueness of the solution (Y, Z, η) .

The rest of this paper is organized as follows. In section 2, we introduce the notations and definitions. In section 3, we give some kind of apriori estimates. In section 4, under a very strong condition, we get the G-martingale representation theorem and the existence and uniqueness results of the solutions of G-BSDEs.

2 Preliminaries

For any $n \times d$ dimensional matrices $\gamma, \tilde{\gamma}$, define

$$\gamma : \tilde{\gamma} := \text{tr}(\gamma^* \tilde{\gamma}), |\gamma| := \sqrt{\gamma : \gamma},$$

where γ^* denotes the transpose of γ .

For a dimension d , let \mathbb{R}^d , \mathbb{S}^d , and \mathbb{D}^d denote the sets of d -dimensional column vectors, $d \times d$ -symmetric matrices, and $d \times d$ -diagonal matrices respectively. For $\sigma_1, \sigma_2 \in \mathbb{S}^d$, $\sigma_1 \leq \sigma_2$ (resp. $\sigma_1 < \sigma_2$) means that $\sigma_2 - \sigma_1$ is nonnegative (resp. positive) definite, and we denote by $[\sigma_1, \sigma_2]$ the set of $\sigma \in \mathbb{S}^d$ satisfying $\sigma_1 \leq \sigma \leq \sigma_2$. For $\xi^1, \xi^2 \in \mathbb{R}^d$, $\xi^1 \leq \xi^2$ means that each element of ξ^1 is less or equal to that of ξ^2 , that is $\xi_i^1 \leq \xi_i^2, i = 1, \dots, d$. We use $\mathbf{0}$ to denote the d -dimensional zero vector or zero matrix, and I_d the $d \times d$ identity matrix. And

for $\gamma, \tilde{\gamma} \in \mathbb{S}^d$, we have

$$|\gamma : \tilde{\gamma}| \leq |\gamma| |\tilde{\gamma}|, \text{ and } -\gamma \leq \tilde{\gamma} \leq \gamma \text{ implies that } |\tilde{\gamma}| \leq |\gamma|.$$

Let $\mathbb{R}^{n \times d \times d}$ denote all $\eta = (\eta^1, \dots, \eta^n)^*$ with $\eta^i, i = 1, \dots, n$ being $d \times d$ matrices. When $\eta^i, i = 1, \dots, n$ are symmetric matrices, we use $\mathbb{S}^{n \times d \times d}$ instead of $\mathbb{R}^{n \times d \times d}$, when $\eta^i = \text{diag}(\eta^{i:1}, \dots, \eta^{i:d}), i = 1, \dots, n$ are diagonal matrices, we use $\mathbb{D}^{n \times d \times d}$ instead of $\mathbb{S}^{n \times d \times d}$, and when $n = 1$, we use \mathbb{D}^d instead of $\mathbb{D}^{1 \times d \times d}$. For any symmetric matrix γ , define

$$\eta : \gamma = (\eta^1 : \gamma, \dots, \eta^n : \gamma)^*,$$

with $\eta^i : \gamma = \text{tr}((\eta^i)^* \gamma), i = 1, \dots, n$. Now we define an operator \cdot , such that

$$\eta \cdot \theta = \sum_{i=1}^n (\eta^i)^* \theta^i, \text{ if } \eta, \theta \in \mathbb{R}^{n \times d \times d},$$

$$\xi \cdot \eta = \eta \cdot \xi = \sum_{i=1}^n \xi^i \eta^i, \text{ if } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^{n \times d \times d},$$

and

$$\xi \cdot \eta : \gamma = \sum_{i=1}^n \xi^i \eta^i : \gamma, \text{ if } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^{n \times d \times d}, \gamma \in \mathbb{S}^d.$$

Define

$$|\eta| = \sqrt{\sum_{i=1}^n \eta^i : \eta^i}, \text{ if } \eta \in \mathbb{R}^{n \times d \times d},$$

and $G(\eta) = (G(\eta^1), \dots, G(\eta^n))^*$ where

$$G(\eta^i) := \frac{1}{2} \sup_{\sigma^2 \in [\underline{\sigma}^2, \bar{\sigma}^2]} (\sigma^2 : \eta^i).$$

Let $\Omega = C([0, \infty], \mathbb{R}^d)$, $\mathcal{F} = \mathcal{B}(\Omega)$. G-expectation \mathbb{E} is a sublinear expectation on the canonical space Ω such that the canonical process B is a G-Brownian motion. We assume the increment $B_{t+s} - B_t$ is $N(\{0\} \times \Sigma s)$ -distributed, for each $t, s \geq 0$, where Σ is a bounded, convex and closed subset of \mathbb{S}^d .

Define

$$G(A) := \frac{1}{2} \mathbb{E}[\langle AB_1, B_1 \rangle] = \frac{1}{2} \sup_{\sigma^2 \in \Sigma} \text{tr}[A \sigma^2], A \in \mathbb{S}^d.$$

We assume $B_t = (B_t^1, \dots, B_t^d)^*$ satisfies that for each fixed t , $B_t^i, i = 2, \dots, n$ is independent from B_t^{i-1}, \dots, B_t^1 . Then it is easy to prove that the matrices in Σ will be diagonal matrices, i.e., any $\sigma^2 \in \Sigma$, $\sigma^2 = \text{diag}(\sigma_{11}^2, \dots, \sigma_{nn}^2)$, with $\sigma_{ii}^2 \in [\underline{\sigma}_{ii}^2, \bar{\sigma}_{ii}^2]$, where $\underline{\sigma}_{ii}^2 = -\mathbb{E}[-B_1^i B_1^i], \bar{\sigma}_{ii}^2 = \mathbb{E}[B_1^i B_1^i]$. In the following, we denote $\underline{\sigma}^2 = \text{diag}(\underline{\sigma}_{11}^2, \dots, \underline{\sigma}_{nn}^2)$, and $\bar{\sigma}^2 = \text{diag}(\bar{\sigma}_{11}^2, \dots, \bar{\sigma}_{nn}^2)$.

Assume \tilde{B} is another G-Brownian motion without the independence assumption, then $\tilde{B}_{t+s} - \tilde{B}_t \sim N(\{0\} \times \tilde{\Sigma} s)$ where $\tilde{\Sigma}$ is a bounded, convex and closed subset of \mathbb{S}^d . Define

$$\tilde{G}(A) := \frac{1}{2} \tilde{\mathbb{E}}[\langle A \tilde{B}_1, \tilde{B}_1 \rangle] = \frac{1}{2} \sup_{Q \in \tilde{\Sigma}} \text{tr}[A Q], A \in \mathbb{S}^d.$$

Denote $Q = (q_{ij}), \underline{Q} = (\underline{q}_{ij}), \bar{Q} = (\bar{q}_{ij})$, where $\underline{q}_{ij} = \inf q_{ij} = -\mathbb{E}[-\tilde{B}_1^i \tilde{B}_1^j], \bar{q}_{ij} = \sup q_{ij} = \mathbb{E}[\tilde{B}_1^i \tilde{B}_1^j]$.

Denote $X_i = B_1^i, i = 1, \dots, d$ and $X = (X_1, \dots, X_d)^*$. Let P be a matrix, and let $Y = PX$, then

$$\begin{aligned} \frac{1}{2} \mathbb{E}[\langle AY, Y \rangle] &= \frac{1}{2} \mathbb{E}[\langle APX, PX \rangle] = \frac{1}{2} \mathbb{E}[\langle P^* APX, X \rangle] \\ &= \frac{1}{2} \sup_{\sigma^2 \in \Sigma} \text{tr}(P^* AP \sigma^2) = G(P^* AP). \end{aligned}$$

We let the elements in matrix $A = (a_{ij})$ take the following values. For given $i, j = 1, \dots, d$, we let $a_{ij} = a_{ji} = 1$, $a_{kl} = 0$, if $k, l = 1, \dots, n$, $(k, l) \neq (i, j)$ and (j, i) , and then

$$\sup_{\sigma^2 \in \Sigma} \sum_{l=1}^n (p_{il}p_{jl} + p_{jl}p_{il})\sigma_{ll}^2 = 2\mathbb{E}[Y_i Y_j], i, j = 1, \dots, d. \quad (2.1)$$

If we let the 1 in above procedure be replaced by -1 , then we get

$$\inf_{\sigma^2 \in \Sigma^2} \sum_{l=1}^n (p_{il}p_{jl} + p_{jl}p_{il})\sigma_{ll}^2 = -2\mathbb{E}[-Y_i Y_j], i, j = 1, \dots, d. \quad (2.2)$$

So as long as there exist matrix P and diagonal matrices set Σ such that (2.1) and (2.2) hold, we can construct random vector $\tilde{B}_1 = Y = PX$ whose covariance matrices set is $\tilde{\Sigma}$ such that any $Q = (q_{ij}) \in \tilde{\Sigma}$, $\underline{q}_{ij} \leq q_{ij} \leq \bar{q}_{ij}$ where $\underline{q}_{ij} = \inf q_{ij} = -\mathbb{E}[-\tilde{B}_1^i \tilde{B}_1^j]$, $\bar{q}_{ij} = \sup q_{ij} = \mathbb{E}[\tilde{B}_1^i \tilde{B}_1^j]$. We denote $\underline{Q} = (\underline{q}_{ij})$, $\bar{Q} = (\bar{q}_{ij})$.

So in this paper we assume for each fixed t , $B_t^i, i = 2, \dots, d$ is independent from B_t^{i-1}, \dots, B_t^1 . Hence Σ is bounded, convex and closed subset of diagonal matrices \mathbb{D}^d .

Let $\langle B \rangle$ denote the quadratic variation of B such that $BB^* - \langle B \rangle$ is a G-martingale. Since for each fixed t , $B_t^i, i = 2, \dots, d$ is independent from B_t^{i-1}, \dots, B_t^1 , we can let $\langle B \rangle_t$ be a diagonal matrix for every t .

For each fixed $T \geq 0$, let

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, B_{t_1}, \dots, B_{t_n}) : \forall n \geq 1, t_1, \dots, t_n \in [0, T], \forall \varphi \in C_{l, Lip}(\mathbb{R}^{d \times n})\}.$$

For fixed $p \geq 1$, define a norm on $L_{ip}(\Omega_T)$

$$\|\xi\|_{L_G^p(\Omega_T)}^p = (\mathbb{E}[|\xi|^p])^{\frac{1}{p}},$$

and denote $L_G^p(\Omega_T)$ be the closure of $L_{ip}(\Omega_T)$ under the norm $\|\cdot\|_{L_G^p(\Omega_T)}$.

\mathbb{L}^p denote the Banach space under the norm $\|X\|_p := (\mathbb{E}[|\xi|^p])^{\frac{1}{p}}$.

Denote $M_G^{p,0}$ be the space with appropriate dimension of elementary process, $\theta_t = \sum_{i=0}^{n-1} \theta_{t_i} \mathbf{1}_{[t_i, t_{i+1})}(t)$ with each component of θ_{t_i} being in $L_G^p(\Omega_{t_i})$. Define norm

$$\|\theta\|_{M_G^p}^p = \mathbb{E}\left[\int_0^T |\theta_t|^p dt\right], \theta \in M_G^{p,0},$$

and let M_G^p denote the closure of $M_G^{p,0}$ under the norm $\|\cdot\|_{M_G^p}^p$.

In the following, denote $M_G^p(\mathbb{R}^d)$ (respectively, $M_G^p(\mathbb{R}^{d \times n})$ and $M_G^p(\mathbb{R}^{n \times d \times d})$) the complete normed space under the norm $\|\cdot\|_{M_G^p}$ with \mathbb{R}^d (respectively, $\mathbb{R}^{d \times n}$ and $\mathbb{R}^{n \times d \times d}$)-valued processes.

For $\beta > 0$ and $\eta \in M_G^2$, we let $M_G^{2,\beta}$ denote the space M_G^2 endowed with the norm

$$\|\eta\|_{M_G^{2,\beta}} := \left\{ \mathbb{E}\left[\int_0^T e^{\beta t} |\eta_t|^2 dt\right] \right\}^{\frac{1}{2}}.$$

It is proved in Denis, Hu and Peng [2] that there exists a weakly compact family \mathcal{P} of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E_P[X], \text{ for } X \in L_{ip}(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

The natural choquet capacity is defined as

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \text{ for } A \in \mathcal{B}(\Omega).$$

DEFINITION 2.1. *A set A is polar if $c(A) = 0$ and a property holds “quasi-surely” (q.s.) if it holds outside a polar set.*

Let's denote $\underline{\sigma}_{min}^2 = \min_{1 \leq i \leq d} \underline{\sigma}_i^2$, and $\bar{\sigma}_{max}^2 = \max_{1 \leq i \leq d} \bar{\sigma}_i^2$.

3 Some Apriori Estimates of G-BSDEs

Consider the following G-BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \eta_s) ds + \int_t^T g(s, Y_s, Z_s, \eta_s) : d\langle B \rangle_s - \int_t^T Z_s^* dB_s + \int_t^T G(\eta_s) ds - \frac{1}{2} \int_t^T \eta_s : d\langle B \rangle_s, \quad (3.1)$$

or, equivalently,

$$-dY_t = f(t, Y_t, Z_t, \eta_t) dt + g(t, Y_t, Z_t, \eta_t) : d\langle B \rangle_t - Z_t^* dB_t + G(\eta_t) dt - \frac{1}{2} \eta_t : d\langle B \rangle_t, Y_T = \xi, \quad (3.2)$$

where

The terminal value is an \mathcal{F}_T -measurable random variable, $\xi : \Omega \mapsto \mathbb{R}^n$.

The generator f maps $\Omega \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^{d \times n} \times \mathbb{D}^{n \times d \times d}$ onto \mathbb{R}^n and is $\mathcal{B} \otimes \mathcal{B}^n \otimes \mathcal{B}^{d \times n} \otimes \mathcal{B}^{n \times d \times d}$ -measurable.

The generator g maps $\Omega \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^{d \times n} \times \mathbb{D}^{n \times d \times d}$ onto $\mathbb{D}^{n \times d \times d}$ and is $\mathcal{B} \otimes \mathcal{B}^n \otimes \mathcal{B}^{d \times n} \otimes \mathcal{B}^{n \times d \times d}$ -measurable.

Here \mathcal{B} is the σ -field of predictable sets of $\Omega \times [0, T]$.

Suppose that $\xi \in L_G^2(\Omega_T)$, $f(\cdot, y, z, \eta), g(\cdot, y, z, \eta) \in M_G^2$ for each $y \in \mathbb{R}^n, z \in \mathbb{R}^{d \times n}, \eta \in \mathbb{D}^{n \times d \times d}$, and f, g are uniformly Lipschitz; i.e., there exists $C > 0$ such that for every t

$$|f(\omega, t, y_1, z_1, \eta_1) - f(\omega, t, y_2, z_2, \eta_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + |\eta_1 - \eta_2|), \forall (y_1, z_1, \eta_1), \forall (y_2, z_2, \eta_2),$$

$$|g(\omega, t, y_1, z_1, \eta_1) - g(\omega, t, y_2, z_2, \eta_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + |\eta_1 - \eta_2|), \forall (y_1, z_1, \eta_1), \forall (y_2, z_2, \eta_2).$$

Then we say (ξ, f, g) are standard parameters for the G-BSDEs.

LEMMA 3.1. $\forall \eta \in M_G^1(\mathbb{D}^{n \times d \times d}), \forall 0 \leq t \leq s \leq T$, we have

$$|\int_t^s \eta_r : d\langle B \rangle_r| \leq K \int_t^s |\eta_r| dr, \quad (3.3)$$

$$\int_t^s (\eta_r)^+ : \underline{\sigma}^2 dr - \int_t^s (\eta_r)^- : \bar{\sigma}^2 dr \leq \int_t^s \eta_r : d\langle B \rangle_r \leq \int_t^s (\eta_r)^+ : \bar{\sigma}^2 dr - \int_t^s (\eta_r)^- : \underline{\sigma}^2 dr, \quad (3.4)$$

where K is a constant, $\eta^+ = ((\eta^1)^+, \dots, (\eta^n)^+)^*$ with $(\eta^i)^+ = \text{diag}((\eta_1^i)^+, \dots, (\eta_d^i)^+)$, $i = 1, \dots, n$, and $\eta^- = ((\eta^1)^-, \dots, (\eta^n)^-)^*$ with $(\eta^i)^- = \text{diag}((\eta_1^i)^-, \dots, (\eta_d^i)^-)$, $i = 1, \dots, n$.

Proof. Notice that the terms in (3.4) are vectors, and the inequalities are for every elements of the vectors. It is easy to prove that (3.3) and (3.4) hold for $\eta \in M_G^{1,0}(\mathbb{D}^{n \times d \times d})$. Continuously extend them to the case $\eta \in M_G^1(\mathbb{D}^{n \times d \times d})$ and we get (3.3) and (3.4).

THEOREM 3.2. $\theta, \zeta \in M_G^2$, θ_s, ζ_s are continuous quasi-surely about s , and $\|\theta\|_{M_G^2}, \|\zeta\|_{M_G^2} \neq 0$. Then there exists a sequence of $\beta(i) \rightarrow \infty$, as $i \rightarrow \infty$, such that

$$\lim_{i \rightarrow \infty} \frac{\mathbb{E}[\int_t^T e^{\beta(i)s} \theta_s^2 ds]}{\beta(i) \mathbb{E}[\int_t^T e^{\beta(i)s} \zeta_s^2 ds]} = 0. \quad (3.5)$$

Proof. $\forall \theta \in M_G^2$, there exists $\theta^n \xrightarrow{M_G^2} \theta, n \rightarrow \infty$, with $\theta_s^n = \sum_{i=0}^{N_n-1} \theta_{s_i^n}^n \mathbf{1}_{[s_i^n, s_{i+1}^n)}(s)$, $\theta_{s_i^n}^n \in \mathbb{L}^2(\Omega_{s_i^n})$, $s_0^n = t, s_{N_n}^n = T$. Then

$$\lim_{n \rightarrow \infty} \mathbb{E}[\int_t^T e^{\beta s} (\theta_s^n)^2 ds] = \mathbb{E}[\int_t^T e^{\beta s} \theta_s^2 ds], \forall \beta > 0.$$

And

$$\begin{aligned} & \mathbb{E}[\int_t^T e^{\beta s} (\theta_s^n)^2 ds] \\ &= \mathbb{E}[\int_t^T e^{\beta s} \sum_{i=0}^{N_n-1} (\theta_{s_i^n}^n)^2 \mathbf{1}_{[s_i^n, s_{i+1}^n)}(s) ds] \\ &= \mathbb{E}[\sum_{i=0}^{N_n-1} (\theta_{s_i^n}^n)^2 \int_{s_i^n}^{s_{i+1}^n} e^{\beta s} ds] \\ &\leq \max_{0 \leq i \leq N_n-1} \mathbb{E}(\theta_{s_i^n}^n)^2 \int_t^T e^{\beta s} ds. \end{aligned}$$

Let $C_n = \max_{0 \leq i \leq N_n-1} \mathbb{E}(\theta_{s_i^n}^n)^2$. Since $\|\theta\|_{M_G^2} \neq 0$, there exists n large enough such that $C_n \neq 0$, and

$$\mathbb{E}[\int_t^T e^{\beta s} (\theta_s^n)^2 ds] \leq C_n \int_t^T e^{\beta s} ds.$$

By the same reason, there exists $\bar{\zeta}^n \xrightarrow{M_G^2} \zeta, n \rightarrow \infty$, with $\bar{\zeta}_s^n = \sum_{i=0}^{M_n-1} \bar{\zeta}_{s_i^n}^n \mathbf{1}_{[s_i^n, s_{i+1}^n)}(s)$, $\bar{\zeta}_{s_i^n}^n \in \mathbb{L}^2(\Omega_{s_i^n})$, $s_0^n = t, s_{M_n}^n = T$, such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[\int_t^T e^{\beta s} (\bar{\zeta}_s^n)^2 ds] = \mathbb{E}[\int_t^T e^{\beta s} \zeta_s^2 ds], \forall \beta > 0.$$

Let $(\zeta_s^n)^2 = \sum_{i=0}^{M_n-1} (\bar{\zeta}_{s_i^n}^n)^2 \mathbf{1}_{[s_i^n, s_{i+1}^n)}(s) + \frac{1}{n}$. Then $(\zeta_s^n)^2 \geq \frac{1}{n}$, $n = 1, 2, \dots$, and

$$\lim_{n \rightarrow \infty} \mathbb{E}[\int_t^T e^{\beta s} (\zeta_s^n)^2 ds] = \mathbb{E}[\int_t^T e^{\beta s} \zeta_s^2 ds], \forall \beta > 0.$$

Let $D_n = -\max_{0 \leq i \leq M_n-1} \{\mathbb{E}[-(\zeta_{s_i^n}^n)^2]\}$. Then $0 < D_n < \infty$, and

$$\begin{aligned} & D_n \int_t^T e^{\beta s} ds - \mathbb{E}[\int_t^T e^{\beta s} (\zeta_s^n)^2 ds] \\ & \leq \mathbb{E}[D_n \sum_{i=0}^{M_n-1} \int_{s_i^n}^{s_{i+1}^n} e^{\beta s} ds - \sum_{i=0}^{M_n-1} (\zeta_{s_i^n}^n)^2 \int_{s_i^n}^{s_{i+1}^n} e^{\beta s} ds] \\ & \leq \sum_{i=0}^{M_n-1} [D_n + \mathbb{E}[-(\zeta_{s_i^n}^n)^2] \int_{s_i^n}^{s_{i+1}^n} e^{\beta s} ds] \\ & \leq 0. \end{aligned}$$

Hence

$$\mathbb{E}[\int_t^T e^{\beta s} (\zeta_s^n)^2 ds] \geq D_n \int_t^T e^{\beta s} ds, n = 1, 2, \dots$$

Let

$$B_n^\beta = \frac{\mathbb{E}[\int_t^T e^{\beta s} (\theta_s^n)^2 ds]}{\beta \mathbb{E}[\int_t^T e^{\beta s} (\zeta_s^n)^2 ds]}.$$

Then

$$B_n^\beta \leq \frac{C_n \int_t^T e^{\beta s} ds}{\beta D_n \int_t^T e^{\beta s} ds} = \frac{C_n}{\beta D_n}.$$

For any $n \in \mathbb{N}$ such that $C_n \neq 0$, let

$$\beta(n) = n \frac{C_n}{D_n} > 0, \quad (3.6)$$

then

$$B_n = \frac{\mathbb{E}[\int_t^T e^{\beta(n)s} (\theta_s^n)^2 ds]}{\beta(n) \mathbb{E}[\int_t^T e^{\beta(n)s} (\zeta_s^n)^2 ds]} \leq \frac{1}{n}. \quad (3.7)$$

Denote

$$\begin{aligned} T_n &= \frac{\mathbb{E}[\int_t^T e^{\beta(n)s} \theta_s^2 ds]}{\beta(n) \mathbb{E}[\int_t^T e^{\beta(n)s} \zeta_s^2 ds]}, \\ l_n &= \frac{\mathbb{E}[\int_t^T e^{\beta(n)s} (\theta_s^n)^2 ds]}{\mathbb{E}[\int_t^T e^{\beta(n)s} \theta_s^2 ds]}, \\ m_n &= \frac{\mathbb{E}[\int_t^T e^{\beta(n)s} (\zeta_s^n)^2 ds]}{\mathbb{E}[\int_t^T e^{\beta(n)s} \zeta_s^2 ds]}. \end{aligned}$$

Then

$$T_n = \frac{m_n}{l_n} B_n. \quad (3.8)$$

We say $l_n, n = 1, 2, \dots$ is bounded. Otherwise there exists subsequence $l_{n_i} \rightarrow \infty, i \rightarrow \infty$, which means $\frac{\mathbb{E}[\int_t^T e^{\beta(n_i)s} (\theta_s^{n_i})^2 ds]}{\mathbb{E}[\int_t^T e^{\beta(n_i)s} \theta_s^2 ds]} \rightarrow \infty, i \rightarrow \infty$. Then for any $M > 1$, there exist $I > 0$, such

that when $i > I$,

$$\mathbb{E}[\int_t^T e^{\beta(n_i)s} (\theta_s^{n_i})^2 ds] > M \mathbb{E}[\int_t^T e^{\beta(n_i)s} \theta_s^2 ds] > 0.$$

Let $\bar{\theta}_s^{n_i} = \sum_{i=0}^{N_n-1} \theta_{s_i^n} \mathbf{1}_{[s_i^n, s_{i+1}^n)}(s)$. Then $\bar{\theta}_s^{n_i}$ and $\theta_s^{n_i}$ are measurable on product measurable space $([t, T] \times \Omega, \mathcal{B}[t, T] \times \mathcal{F})$, since $\mathbf{1}_{[s_i^n, s_{i+1}^n)}(s)$, $\theta_{s_i^n}$ and $\theta_{s_i^n}^n$ are measurable on $\mathcal{B}[t, T] \times \mathcal{F}$. Because θ_s is continuous quasi-surely about s , $\lim_{i \rightarrow \infty} \bar{\theta}_s^{n_i}(\omega) = \theta_s(\omega)$, $q.s.$, and $\theta_s(\omega)$ is also $\mathcal{B}[t, T] \times \mathcal{F}$ measurable.

For $i > I$, let $E_{M,i}^1 \times E_{M,i}^2 = \{(s, \omega) : |\theta_s^{n_i}(\omega)| > \sqrt{M} |\theta_s(\omega)|\}$. Then

$$\mathbb{E}[\int_t^T (\theta_s^{n_i} - \theta_s)^2 ds] > (1 - \frac{1}{\sqrt{M}})^2 \mathbb{E}[\int_{E_{M,i}^1} \mathbf{1}_{E_{M,i}^2} (\theta_s^{n_i})^2 ds] = (1 - \frac{1}{\sqrt{M}})^2 \mathbb{E}[\mathbf{1}_{E_{M,i}^2} \int_{E_{M,i}^1} (\theta_s^{n_i})^2 ds].$$

While $\theta^n \xrightarrow{M_G^2} \theta$, $n \rightarrow \infty$, hence $\mathbb{E}[\mathbf{1}_{E_{M,i}^2} \int_{E_{M,i}^1} (\theta_s^{n_i})^2 ds] \rightarrow 0$, $i \rightarrow \infty$, which means $\mu(E_{M,i}^1)c(E_{M,i}^2) \rightarrow 0$, $i \rightarrow \infty$, where μ is the Borel measure and c is the Choquet capacity defined by $c(A) = \sup_{P \in \mathcal{P}} P(A)$, for $A \in \mathcal{F}$. So

$$\mathbb{E}[\int_t^T e^{\beta(n_i)s} (\theta_s^{n_i})^2 ds] > M \mathbb{E}[\int_t^T e^{\beta(n_i)s} \theta_s^2 ds], \text{ for all } i > I$$

is impossible.

Similarly, we can prove any convergent subsequence l_{n_i} , $\frac{1}{l_{n_i}} \nrightarrow \infty$, $i \rightarrow \infty$, therefore $l_{n_i} \nrightarrow 0$, $i \rightarrow \infty$. And m_n is bounded by the same reason.

Since l_n , $n = 1, 2, \dots$ is bounded, there exists convergent subsequence. Let n_{k_i} , $i = 1, 2, \dots$ be a subsequence of n_k , $k = 1, 2, \dots$, such that $l_{n_{k_i}} \rightarrow a \neq 0$, $i \rightarrow \infty$. By (3.8), we have $\lim_{i \rightarrow \infty} T_{n_{k_i}} = 0$.

COROLLARY 3.3. *In the proof of theorem 3.2, $\beta(n)$ can be any real number such that*

$$\beta(n) \geq n \frac{C_n}{D_n} > 0. \quad (3.9)$$

Hence we have

$$\lim_{\beta \rightarrow \infty} \frac{\mathbb{E}[\int_t^T e^{\beta s} \theta_s^2 ds]}{\beta \mathbb{E}[\int_t^T e^{\beta s} \zeta_s^2 ds]} = 0. \quad (3.10)$$

PROPOSITION 3.4. *Let $((\xi^i, f^i, g^i); i = 1, 2)$ be two standard parameters of the G-BSDE (3.2) and (Y^i, Z^i, η^i) be two solutions in space $M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$ satisfying:*

i) $Y_t^i, \eta_t^i, i = 1, 2$ are continuous in t quasi-surely;

ii) If $Y^1 = Y^2$, $t - a.e., \omega - q.s.$, then $\eta^1 = \eta^2$, $t - a.e., \omega - q.s.$.

Put $\delta Y_t = Y_t^1 - Y_t^2$, $\delta Z_t = Z_t^1 - Z_t^2$, $\delta \eta_t = \eta_t^1 - \eta_t^2$, $\delta f_t = f^1(t, Y_t^1, Z_t^1, \eta_t^1) - f^2(t, Y_t^2, Z_t^2, \eta_t^2)$, and $\delta g_t = g^1(t, Y_t^1, Z_t^1, \eta_t^1) - g^2(t, Y_t^2, Z_t^2, \eta_t^2)$. There exist $\beta_0(\delta Y, \delta \eta)$ such that when $\beta \geq \beta_0(\delta Y, \delta \eta)$, it follows that

$$\|\delta Y\|_{M_G^{2,\beta}}^2 \leq \frac{1}{\underline{\sigma}_{min}^2} \left[e^{\beta T} \mathbb{E} |\delta Y_T|^2 + \frac{1}{\mu^2} \|\delta f\|_{M_G^{2,\beta}}^2 + \frac{\bar{\sigma}_{max}^2}{\nu^2} \|\delta g\|_{M_G^{2,\beta}}^2 \right], \quad (3.11)$$

$$\|\delta Z\|_{M_G^{2,\beta}}^2 \leq \frac{3}{\underline{\sigma}_{min}^2} \left[e^{\beta T} \mathbb{E} |\delta Y_T|^2 + \frac{1}{\mu^2} \|\delta f\|_{M_G^{2,\beta}}^2 + \frac{\bar{\sigma}_{max}^2}{\nu^2} \|\delta g\|_{M_G^{2,\beta}}^2 \right], \quad (3.12)$$

$$\| \delta \eta \|_{M_G^{2,\beta}}^2 \leq \frac{1}{\underline{\sigma}_{min}^2} \left[e^{\beta T} \mathbb{E} |\delta Y_T|^2 + \frac{1}{\mu^2} \| \delta f \|_{M_G^{2,\beta}}^2 + \frac{\bar{\sigma}_{max}^2}{\nu^2} \| \delta g \|_{M_G^{2,\beta}}^2 \right]. \quad (3.13)$$

Proof. Let $(Y, Z, \eta) \in M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$ be a solution of (3.2). Then by (3.3), there exists a constant $K > 0$ such that

$$\begin{aligned} |Y_t| \leq & |\xi| + K \int_0^T |f(s, Y_s, Z_s, \eta_s)| ds + K \int_0^T |g(s, Y_s, Z_s, \eta_s)| ds \\ & + \sup_{0 \leq t \leq T} \left| \int_t^T Z_s^* dB_s \right| + K \int_0^T |G(\eta_s)| ds + K \int_0^T |\eta_s| ds \end{aligned}$$

It follows from Burkholder-Davis-Gundy inequalities that there exists constants $0 < k_2 < K_2 < \infty$ such that

$$k_2 \mathbb{E} \left[\int_0^T (Z_s Z_s^*) : d\langle B \rangle_s \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t Z_s^* dB_s \right|^2 \right] \leq K_2 \mathbb{E} \left[\int_0^T (Z_s Z_s^*) : d\langle B \rangle_s \right].$$

Since $\langle B \rangle_t$, $\underline{\sigma}^2$, and $\bar{\sigma}^2$ are diagonal matrices, only the diagonal elements works in the operation $:$, so by (3.4), we have

$$k_2 \mathbb{E} \left[\int_0^T (Z_s Z_s^*) : \underline{\sigma}^2 ds \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t Z_s^* dB_s \right|^2 \right] \leq K_2 \mathbb{E} \left[\int_0^T (Z_s Z_s^*) : \bar{\sigma}^2 ds \right].$$

Hence $\sup_{0 \leq t \leq T} \left| \int_0^t Z_s^* dB_s \right| \in \mathbb{L}^2$, and $\int_0^T |G(\eta_s)| ds, \int_0^T |\eta_s| ds \in L_G^2(\Omega_T)$. Since (ξ, f, g) are standard parameters, $|\xi| + \int_0^T |f(s, Y_s, Z_s, \eta_s)| ds + \int_0^T |g(s, Y_s, Z_s, \eta_s)| ds$ belongs to $L_G^2(\Omega_T)$ too. So we have $\sup_{0 \leq t \leq T} |Y_t| \in \mathbb{L}^2$.

Applying Itô's formula to $e^{\beta s} |\delta Y_s|^2$ (Li and Peng [7]), we have

$$\begin{aligned} & e^{\beta t} |\delta Y_t|^2 + \int_t^T \beta e^{\beta s} |\delta Y_s|^2 ds + \int_t^T e^{\beta s} \delta Z_s \delta Z_s^* : d\langle B \rangle_s \\ &= e^{\beta T} |\delta Y_T|^2 + \int_t^T 2e^{\beta s} \delta Y_s \cdot (f^1(s, Y_s^1, Z_s^1, \eta_s^1) - f^2(s, Y_s^2, Z_s^2, \eta_s^2)) ds \\ & \quad + \int_t^T 2e^{\beta s} \delta Y_s \cdot (g^1(s, Y_s^1, Z_s^1, \eta_s^1) - g^2(s, Y_s^2, Z_s^2, \eta_s^2)) : d\langle B \rangle_s - \int_t^T 2e^{\beta s} \delta Y_s^* \delta Z_s^* dB_s \\ & \quad + \int_t^T 2e^{\beta s} \delta Y_s \cdot (G(\eta_s^1) - G(\eta_s^2)) ds - \int_t^T e^{\beta s} \delta Y_s \cdot \delta \eta_s : d\langle B \rangle_s. \end{aligned} \quad (3.14)$$

Since $\delta Y \in M_G^2(\mathbb{R}^n)$, $\delta Z \in M_G^2(\mathbb{R}^{d \times n})$, $e^{\beta s} \delta Y_s^* \delta Z_s^* \in M_G^1$, the stochastic integral $\int_t^T 2e^{\beta s} \delta Y_s^* \delta Z_s^* dB_s$ is well defined.

If $\mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds] = 0$, then $\int_t^T e^{\beta s} |\delta Y_s|^2 ds = 0$, $\omega - q.s.$, and $\delta Y = 0$, $t - a.e., \omega - q.s.$, by ii), $\mathbb{E}[\int_t^T e^{\beta s} |\delta \eta_s|^2 ds] = 0$. And from (3.14)

$$\mathbb{E} \left[\int_t^T e^{\beta s} \delta Z_s \delta Z_s^* : d\langle B \rangle_s \right] \leq \mathbb{E} [e^{\beta T} |\delta Y_T|^2].$$

Since

$$\underline{\sigma}_{min}^2 \int_t^T e^{\beta s} |\delta Z_s|^2 ds \leq \int_t^T e^{\beta s} \delta Z_s \delta Z_s^* : d\langle B \rangle_s,$$

we have (3.12).

If $\mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds] \neq 0$, for given β , and $(Y^i, \eta^i)_{i=1,2}$, there exists $C(\beta, (Y^i, \eta^i)_{i=1,2})$ such that

$$\begin{aligned} & C(\beta, (Y^i, \eta^i)_{i=1,2}) \\ &= \frac{\mathbb{E}[\int_t^T 2e^{\beta s} \delta Y_s \cdot (G(\eta_s^1) - G(\eta_s^2)) ds - \int_t^T e^{\beta s} \delta Y_s \cdot \delta \eta_s : d\langle B \rangle_s] + \underline{\sigma}_{min}^2 \mathbb{E}[\int_t^T e^{\beta s} |\delta \eta_s|^2 ds]}{\mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds]}, \end{aligned} \quad (3.15)$$

i.e.,

$$\begin{aligned} & \mathbb{E}[\int_t^T 2e^{\beta s} \delta Y_s \cdot (G(\eta_s^1) - G(\eta_s^2)) ds - \int_t^T e^{\beta s} \delta Y_s \cdot \delta \eta_s : d\langle B \rangle_s] \\ &= C(\beta, (Y^i, \eta^i)_{i=1,2}) \mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds] - \underline{\sigma}_{min}^2 \mathbb{E}[\int_t^T e^{\beta s} |\delta \eta_s|^2 ds]. \end{aligned} \quad (3.16)$$

From (3.14), we have for any constants μ, ν ,

$$\begin{aligned} & e^{\beta t} |\delta Y_t|^2 + \int_t^T \beta e^{\beta s} |\delta Y_s|^2 ds + \int_t^T e^{\beta s} \delta Z_s \delta Z_s^* : d\langle B \rangle_s \\ & \leq e^{\beta T} |\delta Y_T|^2 + \mu^2 \int_t^T e^{\beta s} |\delta Y_s|^2 ds + \frac{1}{\mu^2} \int_t^T e^{\beta s} |\delta f_s|^2 ds + \nu^2 \bar{\sigma}_{max}^2 \int_t^T e^{\beta s} |\delta Y_s|^2 ds \\ & \quad + \frac{\bar{\sigma}_{max}^2}{\nu^2} \int_t^T e^{\beta s} |\delta g_s|^2 ds - \int_t^T 2e^{\beta s} \delta Y_s^* \delta Z_s^* dB_s \\ & \quad + \int_t^T 2e^{\beta s} \delta Y_s \cdot (G(\eta_s^1) - G(\eta_s^2)) ds - \int_t^T e^{\beta s} \delta Y_s \cdot \delta \eta_s : d\langle B \rangle_s, \end{aligned} \quad (3.17)$$

and further

$$\begin{aligned} & e^{\beta t} |\delta Y_t|^2 + (\beta - \mu^2 - \nu^2 \bar{\sigma}_{max}^2) \int_t^T e^{\beta s} |\delta Y_s|^2 ds + \int_t^T e^{\beta s} \delta Z_s \delta Z_s^* : d\langle B \rangle_s \\ & \leq e^{\beta T} |\delta Y_T|^2 + \frac{1}{\mu^2} \int_t^T e^{\beta s} |\delta f_s|^2 ds + \frac{\bar{\sigma}_{max}^2}{\nu^2} \int_t^T e^{\beta s} |\delta g_s|^2 ds - \int_t^T 2e^{\beta s} \delta Y_s^* \delta Z_s^* dB_s \\ & \quad + \int_t^T 2e^{\beta s} \delta Y_s \cdot (G(\eta_s^1) - G(\eta_s^2)) ds - \int_t^T e^{\beta s} \delta Y_s \cdot \delta \eta_s : d\langle B \rangle_s. \end{aligned} \quad (3.18)$$

Hence

$$\begin{aligned} & (\beta - \mu^2 - \nu^2 \bar{\sigma}_{max}^2) \mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds] \\ & \leq e^{\beta T} \mathbb{E}[|\delta Y_T|^2] + \frac{1}{\mu^2} \mathbb{E}[\int_t^T e^{\beta s} |\delta f_s|^2 ds] + \frac{\bar{\sigma}_{max}^2}{\nu^2} \mathbb{E}[\int_t^T e^{\beta s} |\delta g_s|^2 ds] \\ & \quad + C(\beta, (Y^i, \eta^i)_{i=1,2}) \mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds] - \underline{\sigma}_{min}^2 \mathbb{E}[\int_t^T e^{\beta s} |\delta \eta_s|^2 ds]. \end{aligned} \quad (3.19)$$

And then

$$\begin{aligned} & (\beta - \mu^2 - \nu^2 \bar{\sigma}_{max}^2 - C(\beta, (Y^i, \eta^i)_{i=1,2})) \mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds] + \underline{\sigma}_{min}^2 \mathbb{E}[\int_t^T e^{\beta s} |\delta \eta_s|^2 ds] \\ & \leq e^{\beta T} \mathbb{E}[|\delta Y_T|^2] + \frac{1}{\mu^2} \mathbb{E}[\int_t^T e^{\beta s} |\delta f_s|^2 ds] + \frac{\bar{\sigma}_{max}^2}{\nu^2} \mathbb{E}[\int_t^T e^{\beta s} |\delta g_s|^2 ds]. \end{aligned} \quad (3.20)$$

By (3.15),

$$\begin{aligned} & |C(\beta, (Y^i, \eta^i)_{i=1,2})| \\ & \leq \frac{\frac{3}{2}\bar{\sigma}_{max}^2 \mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds] + \frac{5}{2}\bar{\sigma}_{max}^2 \mathbb{E}[\int_t^T e^{\beta s} |\delta \eta_s|^2 ds]}{\mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds]} = \frac{3\bar{\sigma}_{max}^2}{2} + \frac{5\bar{\sigma}_{max}^2}{2} \frac{\mathbb{E}[\int_t^T e^{\beta s} |\delta \eta_s|^2 ds]}{\mathbb{E}[\int_t^T e^{\beta s} |\delta Y_s|^2 ds]}. \end{aligned} \quad (3.21)$$

By corollary 3.3,

$$\lim_{\beta \rightarrow \infty} \frac{\mathbb{E}[\int_t^T e^{\beta s} \delta \eta_s^2 ds]}{\beta \mathbb{E}[\int_t^T e^{\beta s} \delta Y_s^2 ds]} = 0,$$

so we can always choose $\beta_0(\delta Y, \delta \eta)$ large enough such that when $\beta \geq \beta_0(\delta Y, \delta \eta)$, for any given μ, ν , $\beta - \mu^2 - \nu^2 \bar{\sigma}_{max}^2 - C(\beta, (Y^i, \eta^i)_{i=1,2}) \geq \underline{\sigma}_{min}^2$. By (3.20), we get (3.11) and (3.13).

By (3.18), if $C(\beta, (Y^i, \eta^i)_{i=1,2}) \leq 0$, obviously we have (3.12).

Otherwise, it is easy to test that

$$\begin{aligned} & \int_t^T 2e^{\beta s} \delta Y_s \cdot (G(\eta_s^1) - G(\eta_s^2)) ds - \int_t^T e^{\beta s} \delta Y_s \cdot \delta \eta_s : d\langle B \rangle_s \\ & \leq 5 \frac{\bar{\sigma}_{max}^4}{\underline{\sigma}_{min}^2} \int_t^T e^{\beta s} |\delta Y_s|^2 ds + 2\underline{\sigma}_{min}^2 \int_t^T e^{\beta s} |\delta \eta_s|^2 ds, \end{aligned}$$

so by (3.18), we have

$$\begin{aligned} & e^{\beta t} |\delta Y_t|^2 + (\beta - \mu^2 - \nu^2 \bar{\sigma}_{max}^2) \int_t^T e^{\beta s} |\delta Y_s|^2 ds + \int_t^T e^{\beta s} \delta Z_s \delta Z_s^* : d\langle B \rangle_s \\ & \leq e^{\beta T} |\delta Y_T|^2 + \frac{1}{\mu^2} \int_t^T e^{\beta s} |\delta f_s|^2 ds + \frac{\bar{\sigma}_{max}^2}{\nu^2} \int_t^T e^{\beta s} |\delta g_s|^2 ds - \int_t^T 2e^{\beta s} \delta Y_s^* \delta Z_s^* dB_s \\ & \quad + 5 \frac{\bar{\sigma}_{max}^4}{\underline{\sigma}_{min}^2} \int_t^T e^{\beta s} |\delta Y_s|^2 ds + 2\underline{\sigma}_{min}^2 \int_t^T e^{\beta s} |\delta \eta_s|^2 ds, \end{aligned} \quad (3.22)$$

and then

$$\begin{aligned} & e^{\beta t} |\delta Y_t|^2 + (\beta - \mu^2 - \nu^2 \bar{\sigma}_{max}^2 - 5 \frac{\bar{\sigma}_{max}^4}{\underline{\sigma}_{min}^2}) \int_t^T e^{\beta s} |\delta Y_s|^2 ds + \int_t^T e^{\beta s} \delta Z_s \delta Z_s^* : d\langle B \rangle_s \\ & \leq e^{\beta T} |\delta Y_T|^2 + \frac{1}{\mu^2} \int_t^T e^{\beta s} |\delta f_s|^2 ds + \frac{\bar{\sigma}_{max}^2}{\nu^2} \int_t^T e^{\beta s} |\delta g_s|^2 ds - \int_t^T 2e^{\beta s} \delta Y_s^* \delta Z_s^* dB_s \\ & \quad + 2\underline{\sigma}_{min}^2 \int_t^T e^{\beta s} |\delta \eta_s|^2 ds. \end{aligned} \quad (3.23)$$

We choose β large enough such that $\beta - \mu^2 - \nu^2 \bar{\sigma}_{max}^2 - 5 \frac{\bar{\sigma}_{max}^4}{\underline{\sigma}_{min}^2} > 0$, then we have (3.12).

REMARK 3.5.

By the proof of theorem 3.4, we also have

$$\mathbb{E}[\sup_{0 \leq t \leq T} |\delta Y_t|^2] \leq \mathbb{E}[\sup_{0 \leq t \leq T} e^{\beta t} |\delta Y_t|^2] \leq 3 \left[e^{\beta T} \mathbb{E}(|\delta Y_T|)^2 + \frac{1}{\mu^2} \|\delta f\|_{M_G^{2,\beta}}^2 + \frac{\bar{\sigma}_{max}^2}{\nu^2} \|\delta g\|_{M_G^{2,\beta}}^2 \right].$$

REMARK 3.6.

Let $\xi \in L_{ip}(\Omega_T)$, and $M_t = \mathbb{E}_t[\xi]$. by Peng [15], there exist Z, η such that

$$M_t = \mathbb{E}[\xi] + \int_0^t (Z_s)^* dB_s - \left[\int_0^t G(\eta_s) ds - \frac{1}{2} \int_0^t \eta_s : d\langle B \rangle_s \right], \quad (3.24)$$

where $M_t, \eta_t, i = 1, 2, \dots$ are continuous in t quasi-surely, and if $M = 0, t - a.e., \omega - q.s.$, then $\eta = 0, t - a.e., \omega - q.s.$.

Then (M, Z, η) is the solution of G-BSDE

$$M_t = \xi - \int_t^T (Z_s)^* dB_s + \int_t^T G(\eta_s) ds - \frac{1}{2} \int_t^T \eta_s : d\langle B \rangle_s \quad (3.25)$$

with parameter $(\xi, 0, 0)$.

By proposition 3.4, there exist $\beta_0(M, \eta)$ such that when $\beta \geq \beta_0(M, \eta)$

$$\|M\|_{M_G^{2,\beta}}^2 + \|Z\|_{M_G^{2,\beta}}^2 + \|\eta\|_{M_G^{2,\beta}}^2 \leq \frac{5}{\sigma_{min}^2} e^{\beta T} \mathbb{E}[\xi^2]. \quad (3.26)$$

Hence $\beta_0(M, \eta)$ and M, Z, η are uniquely determined by ξ , and we also denote $\beta_0(M, \eta)$ as $\beta_0(\xi)$.

For any $\xi \in L_G^2(\Omega_T)$, there exist $\xi^n \in L_{ip}(\Omega_T)$ such that $\xi^n \xrightarrow{L_G^2} \xi, n \rightarrow \infty$. For each pair n, m , there exist $\beta_0(\xi^n, \xi^m)$ such that when $\beta \geq \beta_0(\xi^n, \xi^m)$, we have

$$\|M^n - M^m\|_{M_G^{2,\beta}}^2 + \|Z^n - Z^m\|_{M_G^{2,\beta}}^2 + \|\eta^n - \eta^m\|_{M_G^{2,\beta}}^2 \leq \frac{5}{\sigma_{min}^2} e^{\beta T} \mathbb{E}[(\xi^n - \xi^m)^2]. \quad (3.27)$$

Define

$$\begin{aligned} \mathcal{L}_G^2(\Omega_T) &= \{\xi \in L_G^2(\Omega_T) : \text{there exist } \xi^n \in L_{ip}(\Omega_T) \text{ and } \beta < \infty \text{ such that} \\ &\quad \xi^n \xrightarrow{L_G^2} \xi, n \rightarrow \infty \text{ and } \beta_0(\xi^n, \xi^m) \leq \beta, n, m = 1, 2, \dots\} \end{aligned} \quad (3.28)$$

4 G-martingale representation and existence and uniqueness of G-BSDEs under a strong condition

THEOREM 4.1. *For any $\xi \in \mathcal{L}_G^2(\Omega_T)$, denote $M_t = \mathbb{E}_t[\xi]$, then there exist unique $(Z, \eta) \in M_G^2(\mathbb{R}^d) \times M_G^2(\mathbb{D}^{d \times d})$ such that*

$$M_t = M_0 + \int_0^t Z_s^* dB_s - \left[\int_0^t G(\eta_s) ds - \frac{1}{2} \int_0^t \eta_s : d\langle B \rangle_s \right], \quad (4.1)$$

Proof. For any $\xi \in \mathcal{L}_G^2(\Omega_T)$, there exist $\xi^n \in L_{ip}(\Omega_T)$ and $\beta < \infty$ such that $\xi^n \xrightarrow{L_G^2} \xi, n \rightarrow \infty$ and $\beta_0(\xi^n, \xi^m) \leq \beta, n = 1, 2, \dots$. For every ξ^n , by Peng [15], there exist M^n, Z^n, η^n such that

$$M_t^n = \mathbb{E}[\xi^n] + \int_0^t (Z_s^n)^* dB_s - \left[\int_0^t G(\eta_s^n) ds - \frac{1}{2} \int_0^t \eta_s^n : d\langle B \rangle_s \right], \quad (4.2)$$

$M_t^n, \eta_t^n, i = 1, 2, \dots$ are continuous in t quasi-surely, and if $M^m = M^n, t - a.e., \omega - q.s.$, then $\eta^m = \eta^n, t - a.e., \omega - q.s.$.

Then (M^n, Z^n, η^n) is the solution of G-BSDE

$$M_t^n = \xi^n - \int_t^T (Z_s^n)^* dB_s + \int_t^T G(\eta_s^n) ds - \frac{1}{2} \int_t^T \eta_s^n : d\langle B \rangle_s \quad (4.3)$$

with parameter $(\xi^n, 0, 0)$.

By proposition 3.4, we have when $\bar{\beta} \geq \beta(\xi^m, \xi^n)$

$$\|M^m - M^n\|_{M_G^{2,\bar{\beta}}}^2 + \|Z^m - Z^n\|_{M_G^{2,\bar{\beta}}}^2 + \|\eta^m - \eta^n\|_{M_G^{2,\bar{\beta}}}^2 \leq e^{\bar{\beta}T} \frac{5}{\sigma_{min}^2} \|\xi^m - \xi^n\|_{L_G^2}^2,$$

and consequently,

$$\|M^m - M^n\|_{M_G^2}^2 + \|Z^m - Z^n\|_{M_G^2}^2 + \|\eta^m - \eta^n\|_{M_G^2}^2 \leq e^{\bar{\beta}T} \frac{5}{\sigma_{min}^2} \|\xi^m - \xi^n\|_{L_G^2}^2. \quad (4.4)$$

Let $\bar{\beta} = \beta$, then (4.4) holds for the constant β and $m, n = 1, 2, \dots$, and (M^n, Z^n, η^n) is a Cauchy sequence in M_G^2 , so there exist (M, Z, η) such that $(M^n, Z^n, \eta^n) \xrightarrow{M_G^2} (M, Z, \eta)$. Since

$$\begin{aligned} & \mathbb{E} \left(\int_t^T G(\eta_s^n) ds - \int_t^T G(\eta_s) ds \right)^2 \\ & \leq \mathbb{E} \left(\int_t^T |G(\eta_s^n) - G(\eta_s)| ds \right)^2 \\ & \leq \bar{\sigma}_{max}^4 \mathbb{E} \left[\int_t^T |\eta_s^n - \eta_s|^2 ds \right] \rightarrow 0, \end{aligned}$$

we have $\int_t^T G(\eta_s^n) ds \xrightarrow{\mathbb{L}_G^2} \int_t^T G(\eta_s) ds$. By Denis, Hu and Peng[2] proposition 17, there exists a subsequence $\int_t^T G(\eta_s^{n_i}) ds \rightarrow \int_t^T G(\eta_s) ds, q.s..$ Similarly, there exist a subsequence $\xi^{n_{i_j}}$ converging to ξ quasi-surely, a subsequence $\int_t^T (Z_s^{n_{i_{j_k}}})^* dB_s$ converging to $\int_t^T Z_s^* dB_s$ quasi-surely, and a subsequence $\int_t^T \eta_s^{n_{i_{j_{k_l}}}} : d\langle B \rangle_s$ converging to $\int_t^T \eta_s : d\langle B \rangle_s$ quasi-surely. For simplicity, we denote the index of the quasi-surely convergent subsequences as k . Then $M_t^k \rightarrow M_t, k \rightarrow \infty, q.s..$

Since (M^n, Z^n, η^n) satisfy (4.1), (M, Z, η) satisfy (4.1), q.s..

The uniqueness of (Z, η) follows from proposition 3.4.

THEOREM 4.2. *Given standard parameters (ξ, f, g) , let $\Psi(y, z, \zeta) = \xi + \int_t^T f(s, y_s, z_s, \zeta_s) ds + \int_t^T g(s, y_s, z_s, \zeta_s) : d\langle B \rangle_s$. Suppose for any $(y, z, \zeta) \in M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$, $\Psi(y, z, \zeta) \in \mathcal{L}_G^2(\Omega_T)$, and there exists $\beta > 0$, for any $(y, z, \zeta), (y', z', \zeta') \in M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$, $\beta_0(\Psi(y, z, \zeta), \Psi(y', z', \zeta')) \leq \beta$, then there exists a unique triplet $(Y, Z, \eta) \in M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$ which solves G-BSDE(3.1) in the sense of \mathcal{P} -q.s., and Y is a \mathcal{P} -q.s. continuous process.*

Proof. Firstly, we prove there exists a mapping from $M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$ into $M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^*(\mathbb{D}^{n \times d \times d})$,

$$\Phi : (y, z, \zeta) \rightarrow (Y, Z, \eta)$$

where (Y, Z, η) is the solution of the G-BSDE(3.1) with generator $f(t, y_t, z_t, \zeta_t), g(t, y_t, z_t, \zeta_t)$, i.e.,

$$Y_t = \xi + \int_t^T f(s, y_s, z_s, \zeta_s) dt + \int_t^T g(s, y_s, z_s, \zeta_s) : d\langle B \rangle_s - \int_t^T Z_s^* dB_s + \int_t^T G(\eta_s) ds - \frac{1}{2} \int_t^T \eta_s : d\langle B \rangle_s.$$

Since $\xi + \int_0^T f(s, y_s, z_s, \zeta_s) dt + \int_0^T g(s, y_s, z_s, \zeta_s) : d\langle B \rangle_s \in \mathcal{L}_G^2 \subseteq L_G^2$, we can define a G-martingale $M_t := \mathbb{E}_t[\xi + \int_0^T f(s, y_s, z_s, \zeta_s) dt + \int_0^T g(s, y_s, z_s, \zeta_s) : d\langle B \rangle_s]$. By theorem 4.1,

there exist unique $(Z, \eta) \in M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$ such that

$$M_t = M_0 + \int_0^t Z_s^* dB_s - \int_0^t G(\eta_s) ds + \frac{1}{2} \int_0^t \eta_s : d\langle B \rangle_s, \mathcal{P} - q.s.$$

Since for every $s \in [0, T]$, $\langle B \rangle_s$ is a diagonal matrix, only the diagonal elements enter the operation $:$, so the uniqueness of η means the diagonal elements is uniquely determined. Hence we choose η to be a diagonal matrix process.

Define the process Y by

$$Y_t = M_t - \int_0^t f(s, y_s, z_s, \eta_s) ds - \int_0^t g(s, y_s, z_s, \eta_s) : d\langle B \rangle_s,$$

which is \mathcal{P} -q.s. continuous by Li and Peng [7]. And Y is also given by

$$Y_t = \mathbb{E}_t[\xi + \int_t^T f(s, y_s, z_s, \zeta_s) ds + \int_t^T g(s, y_s, z_s, \zeta_s) : d\langle B \rangle_s]. \quad (4.5)$$

So

$$\begin{aligned} & Y_t + \int_t^T Z_s^* dB_s - \int_t^T G(\eta_s) ds + \frac{1}{2} \int_t^T \eta_s : d\langle B \rangle_s \\ &= M_0 + \int_0^T Z_s^* dB_s - \int_0^T G(\eta_s) ds + \frac{1}{2} \int_0^T \eta_s : d\langle B \rangle_s - \int_0^t f(s, y_s, z_s, \eta_s) ds - \int_0^t g(s, y_s, z_s, \eta_s) : d\langle B \rangle_s \\ &= M_T - \int_0^t f(s, y_s, z_s, \eta_s) ds - \int_0^t g(s, y_s, z_s, \eta_s) : d\langle B \rangle_s \\ &= \xi + \int_t^T f(s, y_s, z_s, \eta_s) ds + \int_t^T g(s, y_s, z_s, \eta_s) : d\langle B \rangle_s, \mathcal{P} - q.s., \end{aligned}$$

which is

$$Y_t = \xi + \int_t^T f(s, y_s, z_s, \zeta_s) ds + \int_t^T g(s, y_s, z_s, \zeta_s) : d\langle B \rangle_s - \int_t^T Z_s^* dB_s + \int_t^T G(\eta_s) ds - \frac{1}{2} \int_t^T \eta_s : d\langle B \rangle_s.$$

By (4.5), we have $\sup_{0 \leq t \leq T} |Y_t| \in \mathbb{L}^2$.

Let $(y^1, z^1, \zeta^1), (y^2, z^2, \zeta^2)$ be two elements of $M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$, and let (Y^1, Z^1, η^1) and (Y^2, Z^2, η^2) be the associated solutions. Since $f(y, z, \zeta), g(y, z, \zeta)$ do not contain Y, Z, η , applying proposition 3.4,

$$\begin{aligned} & \|\delta Y\|_{M_G^{2,\beta}}^2 + \|\delta Z\|_{M_G^{2,\beta}}^2 + \|\delta \eta\|_{M_G^{2,\beta}}^2 \\ & \leq \frac{5}{\underline{\sigma}_{min}^2 \mu^2} \mathbb{E} \int_0^T e^{\beta s} |f(s, y_s^1, z_s^1, \zeta_s^1) - f(s, y_s^2, z_s^2, \zeta_s^2)|^2 ds \\ & \quad + \frac{5}{\underline{\sigma}_{min}^2 \nu^2} \mathbb{E} \int_0^T e^{\beta s} |g(s, y_s^1, z_s^1, \zeta_s^1) - g(s, y_s^2, z_s^2, \zeta_s^2)|^2 ds. \end{aligned}$$

Since f, g is uniformly Lipschitz in y, z, ζ ,

$$\|\delta Y\|_{M_G^{2,\beta}}^2 + \|\delta Z\|_{M_G^{2,\beta}}^2 + \|\delta \eta\|_{M_G^{2,\beta}}^2 \leq \frac{5K}{\underline{\sigma}_{min}^2} \left(\frac{1}{\mu^2} + \frac{1}{\nu^2} \right) [\|\delta y\|_{M_G^{2,\beta}}^2 + \|\delta z\|_{M_G^{2,\beta}}^2 + \|\delta \zeta\|_{M_G^{2,\beta}}^2],$$

where K is a constant. By the proof of proposition 3.4, we can choose μ, ν large enough such that

$$\frac{5K}{\underline{\sigma}_{min}^2} \left(\frac{1}{\mu^2} + \frac{1}{\nu^2} \right) < 1,$$

and then the mapping Φ is a contraction from $M_G^2(\mathbb{R}^n) \times M_G^2(\mathbb{R}^{d \times n}) \times M_G^2(\mathbb{D}^{n \times d \times d})$ onto itself and there exists a fixed point, which is the unique solution of the G-BSDE.

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